Type Theory

- type = set
- write $e \in A$ as $e : A$
- set theory: $\emptyset$, $\{a, b\}$, $\bigcup$, $\mathcal{P}(A)$, $\bigcap$, $\{x \in A \mid \varphi x\}$
- type theory: an alternative setup, other constructions

Powerset axiom:

$$\forall x \in B.\ x \in A$$

$$\forall A.\ \exists P.\ \forall B.\ (B \in P \Leftrightarrow B \subseteq A)$$

<u>Operation</u>: $\mathcal{P}(A)$

$$\forall B.\ B \in \mathcal{P}(A) \Leftrightarrow B \subseteq A$$

$$\{x \in A \mid \varphi(x)\} \in \mathcal{P}(A)$$

Type theory: operations / constructions $\leftarrow$

+ equations $\leftarrow$

$A \in \mathrm{Set}$ $\qquad$ $A : \mathrm{Type}$

$\uparrow$

class

Binary product

Construction:
$$\frac{A : \mathrm{Type} \qquad B : \mathrm{Type}}{A \times B : \mathrm{Type}}$$
prod A B

Pairing:
$$\frac{a : A \qquad b : B}{(a, b) : A \times B}$$
pair A B

Projections:
$$\frac{u : A \times B}{\pi_1(u) : A} \qquad \frac{u : A \times B}{\pi_2(u) : B}$$

Equations:
$$\pi_1(a, b) = a \qquad (\pi_1(u), \pi_2(u)) = u$$
$$\pi_2(a, b) = b$$

Binary sum

$$\frac{A : \mathrm{Type} \qquad B : \mathrm{Type}}{A + B : \mathrm{Type}}$$

$$\frac{a : A}{\iota_1(a) : A + B} \qquad \frac{b : B \qquad A : \mathrm{Type} \qquad B : \mathrm{Type}}{\iota_2(b) : A + B}$$

$$\frac{u : A + B \qquad x : A \vdash e_1 : C \qquad b : B \vdash e_2 : C}{(\mathrm{case}\ u\ \mathrm{of}\ \iota_1(x) \Rightarrow e_1 \mid \iota_2(b) \Rightarrow e_2) : C}$$

$$\underbrace{x_1 : A_1, x_2 : A_2, \dots, x_n : A_n}_{\text{known variables / parameters}} \vdash e : B$$

CONTEXTS

$e[a/x]$

$$(\text{case } \iota_1(a) \text{ of } \iota_1(x) \Rightarrow e_1 \mid \iota_2(y) \Rightarrow e_2) = e_1[x := a]$$

$$\dots \iota_2(b) \quad \dots\dots \quad = e_2[y := b]$$

$$(\text{case } u \text{ of } \iota_1(x) \Rightarrow \iota_1(x) \mid \iota_2(y) \Rightarrow \iota_2(y)) = u$$

Dependent types / Families

$A : I \to \text{Type}$ family

indexing

dependent type : $A(i)$

" Let $n \in \mathbb{N}$. ......

Consider $\vec{x} \in \mathbb{R}^n$

depends on n.

In the interva $[n, n+2]$ there are ......

Never:

$\mathbb{R}^{\square} : \mathbb{N} \to \text{Set}$

$n \mapsto \mathbb{R}^n$

Dependent product

$$A : I \to \mathrm{Type} \qquad \frac{i : I \vdash A(i) : \mathrm{Type}}{\vdash \prod_{(i:I)} A(i) : \mathrm{Type}}$$

$$\frac{i : I \vdash e(i) : A(i)}{\vdash (\lambda(i:I).\, e(i)) : \prod_{(i:I)} A(i)}$$

$$(i \mapsto e(i))$$

$$\frac{f : \prod_{(i:I)} A(i) \qquad j : I}{f(j) : A(j)}$$

$$(\lambda i.\, e(i))(j) = e(j)$$

$$\prod_{(i:I)} A = I \to A$$

$\forall x.\ x + y = y + x$

$\forall z.\ z + y = y + z$

$\int x^2 dx = \frac{x^3}{3} + C$

$\int t^2 dt \quad \lambda x. \frac{x^3}{3}$

$I = \{0, 1\}$

$A_0 := B$

$A_1 := C$

$$\prod_{i : \{0,1\}} A_i \cong B \times C$$

Dependent sum

$$\sum_{(i:I)} A(i)$$ elements (i,a) where $i:I$ and $a:A(i)$

$\pi_1(i,a) = i$

$\pi_2(i,a) = a$

- $I = \{0,1\}$
  $A_0 := B$
  $A_1 := C$
  $$\sum_{(i:I)} A(i) \cong B + C$$

- $$\sum_{(i:I)} A = I \times A$$

Logic via Sets

$\cancel{\{\bot, \top\}}$

Truth values / propositions = sub-singleton

$$\frac{a:A \qquad b:A}{a = b}$$

Propositions-as-types

S sub-singleton — "true" = has an element

Logic	Types	
$\bot$	$\emptyset$	
$\top$	$\mathbb{1} = \{*\}$	unit type
$p \wedge q$	$p \times q$	
$p \Rightarrow q$	$p \to q$	
$p \vee q$	$\lVert p + q \rVert$	truncation
		$A/_{A \times A}$
$\forall x \in A. p(x)$	$\prod_{(x:A)} p(x)$	
$\exists x \in A. p(x)$	$\lVert \sum_{(x:A)} p(x) \rVert$	
$a \sim b$	?	