① Review

- type theory:
  types, elements, equations

- constructions: $\times \quad + \quad \Pi \quad \Sigma \quad \to$

- logic via types:

  sub-singletons "A true" .... "A has an element"

  <u>Curry-Howard correspondence</u>:

$A \wedge B$	$A \times B$	
$A \Rightarrow B$	$A \to B$	
$A \vee B$	$A + B$	decision (not sub-singleton)
$\leftrightarrow$	$\lVert A + B \rVert$	disjunction
$\forall x \in A . P(x)$	$\Pi(x : A) . P(x)$	
$\exists x \in A . P(x)$	$\Sigma(x : A) . P(x)$	construction
$\leftrightarrow$	$\lVert \Sigma(x : A) . P(x) \rVert$	existental

Equality as type?

② Identity types

Naive attempt: A type, $x, y : A$

$$\mathrm{Id}_A(x,y) := \begin{cases} \emptyset & \text{if } x \neq y \\ \{\mathrm{refl}_A(x)\} & \text{if } x = y \end{cases}$$

Rules:

$$\frac{\vdash a : A \qquad \vdash b : A}{\vdash \mathrm{Id}_A(a,b) \text{ type}} \qquad\qquad \frac{\vdash a : A}{\vdash \mathrm{refl}_A(a) : \mathrm{Id}_A(a,a)}$$

deconstructor: J too scary for this lecture (reading material)

We have: if $a =_A b$ for $a, b : A$ then $\mathrm{refl}_A(a) : \mathrm{Id}_A(a,b)$

$$\frac{\mathrm{refl}_A(a) : \mathrm{Id}_A(a,a) \qquad a =_A b}{\mathrm{refl}_A(a) : \mathrm{Id}_A(a,b)} \quad \text{substitution}$$

How about $$\frac{\vdash p : \mathrm{Id}_A(a,b)}{\vdash a =_A b} \ ? \quad \text{Equality reflection}$$

③ Universes

level of types $\vdash A$ type
$\vdash A \equiv B$

level of elements $\vdash a : A$
$\vdash a \equiv_A b$

Universe = a type whose elements are types

~~Type~~ = the type of all types (including Type)
(paradoxical Girard : every type has an element, including $Id_{\mathbb{N}}(0,1)$)

Set theory :

Set ↑ "small"

class ↑ "large" V --- class of all sets

$Type_0$ the type of all small types

Keep going :

$\mathbb{N}$, Bool : $Type_0$

$Type_0$: $Type_1$

$Type_0 \to Type_0$

$Type_2$

⋮

In general :

$Type_\alpha$

↳ What is this?

Level

$\cancel{\mathrm{Prop}_\alpha}$ .... the universe of sub-singleton types at level α

Prop .... a universe of propositions

In set theory with excluded middle

$$\mathrm{Prop} \cong \{\bot, \top\}$$

④ Natural numbers

$$\frac{}{\vdash \mathbb{N} : \mathrm{Type}_0} \qquad \frac{}{\vdash 0 : \mathbb{N}} \qquad \frac{\vdash e : \mathbb{N}}{\vdash \mathrm{succ}(e) : \mathbb{N}}$$

$$\frac{\begin{array}{l} \vdash A : \mathbb{N} \to \mathrm{Type} \\ \vdash b : A(0) \\ x : \mathbb{N}, y : A(x) \vdash i(x,y) : A(\mathrm{succ}(x)) \\ \vdash e : \mathbb{N} \end{array}}{\vdash \mathrm{ind}_{\mathbb{N}}(A, b, i, e) \; : A(e)}$$

G group

Freely Generated by a, b, c.

$$G \xrightarrow{\varphi} H$$

Need $\varphi(a), \varphi(b), \varphi(c)$

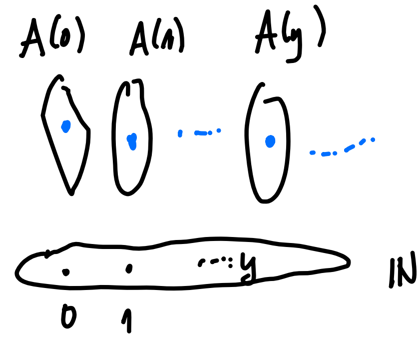
$$\begin{tikzcd} \{a,b,c\} \arrow[r, "\subseteq"] \arrow[dr, "f"'] & G \arrow[d, "\exists! \bar{f}"] \\ & H \end{tikzcd}$$

$$\mathrm{ind}_{\mathbb{N}}(A, b, i, 0) = b$$

$$\mathrm{ind}_{\mathbb{N}}(A, b, i, \mathrm{succ}(e)) = i(e, \mathrm{ind}_{\mathbb{N}}(A, b, i, e))$$

Another way

$$\mathrm{ind}_{\mathbb{N}} : \prod \underbrace{(A : \mathbb{N} \to \mathrm{Type})}_{\text{motive}} . \underbrace{A(0)}_{\text{base}} \to \Big(\underbrace{\prod(x : \mathbb{N}) . A(x) \to A(\mathrm{succ}(x))}_{\text{induction step}}\Big) \to \underbrace{\prod(y : \mathbb{N}) . A(y)}_{\text{Section of } A}$$



5 Invisible mathematics

If $f : X \to Y$ is linear then

$$f(2x+y) = 2 \cdot f(x) + f(y).$$

Guess work:

1) X, Y vector spaces over a field K

2) $x \in X$, $y \in X$

$$f(\overset{1_K +_K 1_K}{2} \cdot_X x +_X y) =_Y 2 \cdot_Y f(x) +_Y f(y)$$

Homework: Natural numbers game