# Inductive types

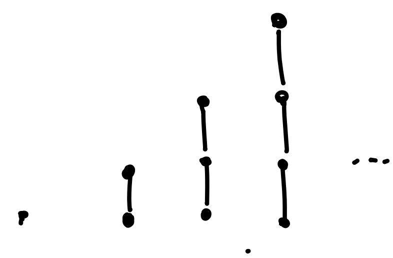
<u>Idea</u>:

- basic objects
- constructors: buid new objects from existing ones

"inductively"

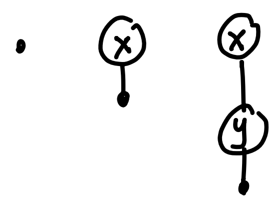
<u>Example 1</u>: natural numbers

- basic object: 0
- Constructor: succ



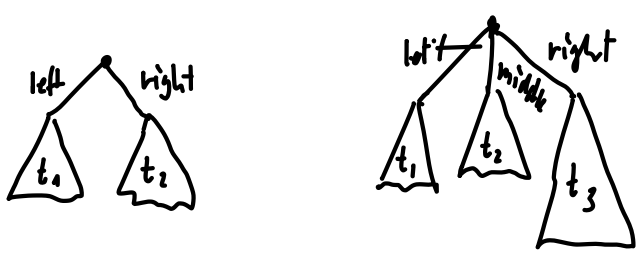
<u>Example</u>: lists

- empty list: [ ]
- "cons": $x :: \ell$ (element, list)



<u>Example</u>: 2-3 trees

- leaf •
- constructors:



# Well-founded trees

**Signature:**

- set $A$ of node kinds
- $B : A \to \mathrm{Set}$ branching

"Inductively" generate $W(A,B)$:

- given any $a \in A$ and $f : B(a) \to W(A,B)$, form $\mathrm{tree}(a,f) \in W(A,B)$

**Example:** $\mathbb{N}$

$A := \{z, s\}$ $\quad B(z) = \emptyset$ $\quad 0_{W(A,B)} : \emptyset \longrightarrow W(A,B)$

$B(s) = \{*\}$

$$
\begin{aligned}
W(A,B) = \{ \\
&\mathrm{tree}(z, 0_{W(A,B)}), \\
&\mathrm{tree}(s, (* \mapsto \mathrm{tree}(z, 0)), \\
&\mathrm{tree}(s, (* \mapsto \mathrm{tree}(s, (* \mapsto \mathrm{tree}(z, 0)))) \\
&\vdots \\
\}
\end{aligned}
$$

## Example:

Let $X$ be a set (of elements).

$$A = \{e\} + X$$

$$B(\mathrm{inl}(e)) := \emptyset$$

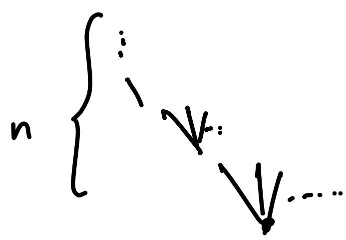
$$B(\mathrm{inr}(x)) := \{*\}$$

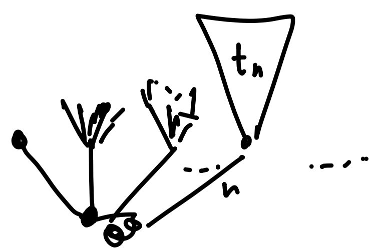
## Example: Countably branching trees $T$

- leaf $\in T$
- given $t : \mathbb{N} \to T$, $\mathrm{tree}(t) \in T$

leaf

$t_n$ full tree of height $n$



What does "inductively generated" mean?

Given a signature $A, B : A \to \mathrm{Set}$ define a functor

$$P_{A,B} : \mathrm{Set} \to \mathrm{Set}$$

<u>polynomial functor</u>

$$P_{A,B}(X) := \sum_{a \in A} X^{B(a)}$$

element: $(a, f : B(a) \to X)$

$$P_{A,B}(f : X \to Y) = ?$$

The set $W(A,B)$ is the <u>initial $P_{A,B}$-algebra</u>.

Recall: A $P_{A,B}$-algebra $(X, h)$ is a set $X$ with a map $h : P_{A,B}(X) \to X$.

Ring $(R, 0, 1, +, \cdot, -)$

$$h : \underset{R^0}{\{z\}} + \underset{R^0}{\{o\}} + R^2 + R^2 + R \longrightarrow R$$

<u>Example</u>: $\mathbb{N}$

$$P_{A,B}(X) = \cdots \cong X^0 + X^1 \cong \mathbb{1} + X$$

$$h : P_{A,B}(X) \to X$$

$$h : \mathbb{1} + X \to X$$

$x \in X \quad g : X \to X \quad$ where $h = [x, g]$

<u>Initial</u>: For every $P_{A,B}$-algebra $\underline{(X, h)}$ there is a unique $r : W(A,B) \to X$ s.t.

$$\begin{tikzcd} P_{A,B}(W(A,B)) \arrow[r, "P_{A,B}(r)"] \arrow[d, "w_{A,B}"'] \arrow[dr, phantom, "="] & P_{A,B}(X) \arrow[d, "h"] \\ W(A,B) \arrow[r, "r"'] & X \end{tikzcd}$$

? $\rightsquigarrow$ $w_{A,B}$

## Inductive types in type theory

Universal property for $\mathbb{N}$:

Given $X$, $x \in X$, $g : X \to X$ there is a unique map $\mathbb{N} \xrightarrow{r} X$ such that

$$\begin{tikzcd} 1 + \mathbb{N} \arrow[r, "1+r"] \arrow[d, "{[0, \mathrm{succ}]}"'] & 1 + X \arrow[d, "{[x, g]}"] \\ \mathbb{N} \arrow[r, "r"'] & X \end{tikzcd}$$

$r(0) = x$

$r(\mathrm{succ}(n)) = g(r(n))$

RECURSION PRINCIPLE (primitive recursion)

## Dependent version

Start with $X : \mathbb{N} \to \mathrm{Set}$

How to construct a map $r : \prod_{(n : \mathbb{N})} X(n)$

Data: $x \in X(0)$

$g : \prod_{(n:\mathbb{N})} X(n) \to X(\mathrm{succ}\, n)$

There is a unique $r : \prod_{(n:\mathbb{N})} X(n)$ s.t.

$$r(0) = x \qquad (1)$$

$$r(\mathrm{succ}\, n) = g\, n\, (r(n)) \qquad (2)$$

$$\mathrm{ind}_{\mathbb{N}}^{X} : \overset{x}{X(0)} \to \overset{g}{\left(\prod_{(n:\mathbb{N})} X(n) \to X(\mathrm{succ}\, n)\right)} \to \overset{r}{\prod_{(n:\mathbb{N})} X(n)}$$

$\parallel$

ind

$$\mathrm{ind}\ x\ g\ 0 = x \qquad (1)$$

$$\mathrm{ind}\ x\ g\ (\mathrm{succ}\, n) = g\ n\ (\mathrm{ind}\ x\ g\ n) \qquad (2)$$

Logical reading: $\varphi : \mathbb{N} \to \mathrm{Prop}$

$$\varphi(0) \Rightarrow \left(\forall_{(n:\mathbb{N})} . \varphi(n) \Rightarrow \varphi(\mathrm{succ}\, n)\right) \Rightarrow \forall_{(m:\mathbb{N})} \varphi(m)$$